Proof of the Branner-Hubbard conjecture on Cantor Julia sets

Weiyuan Qiu Yongcheng Yin

Abstract

By means of a nested sequence of some critical pieces constructed by Kozlovski, Shen, and van Strien, and by using a covering lemma recently proved by Kahn and Lyubich, we prove that the Julia set of a polynomial is a Cantor set if and only if each component of the filled-in Julia set containing critical points is aperiodic. This result was a conjecture raised by Branner and Hubbard in 1992.

1 Introduction

For a complex polynomial f of degree $d \geqslant 2$, the set

$$K_f = \{z \in \mathbb{C} \mid \text{ the sequence } \{f^n(z)\} \text{ is bounded}\}$$

is called the filled-in Julia set of f, where f^n is the n-th iterate of f. The Julia set J_f of f is the boundary of K_f. A component of K_f is called critical if it contains critical points. We denote the component of K_f containing x by $K_f(x)$. A component $K_f(x)$ is aperiodic if $f^n K_f(x) \neq K_f(x)$ for all $n > 0$.

P. Fatou and G. Julia proved the following theorem.

Theorem A ([10] and [14]). *(1) The Julia set of a complex polynomial f is connected if and only if K_f contains all critical points of f.*

(2) The Julia set of a complex polynomial f is a Cantor set if K_f contains no critical points of f.

Fatou conjectured that the condition in Theorem A(2) is also necessary for the Julia set to be a Cantor set. But this was disproved by Brolin in [6]. He gave some real cubic polynomials with Cantor Julia set $J_f = K_f$ containing one critical point.

Using combinatorial system of tableaus, Branner and Hubbard completely settled the question of when the Julia set of a cubic polynomial is a Cantor set. They proved

Theorem B ([4]). *For a cubic polynomial f with one critical point in K_f, the Julia set J_f is a Cantor set if and only if the critical component of K_f is aperiodic.*

The same combinatorics was used by Yoccoz to prove the local connectivity of the Julia set of a quadratic polynomial which has no irrational indifferent periodic points and which is not infinitely renormalizable. Transferring this result to parameter space, he proved that the Mandelbrot set is locally connected at these parameters. See [12] and [27]. Yoccoz introduced a partition of the complex plane by using external rays and equipotential curves. Such partition is called a *Yoccoz puzzle*. It becomes a powerful tool in the study of dynamics of polynomials, see for example [2], [11], [12], [13], [16], [17], [18], [22], [23], [24], [25], [27], [35], and [37]. In [13], Jiang gives the first proof that the Julia set of an unbranched infinitely renormalizable quadratic polynomial having complex bounds is locally connected. A different proof has been given by McMullen in [24]. Other puzzles are used to prove local connectivity of the Julia sets of some quadratic Siegel polynomials and cubic Newton maps, see [29], [30], [31] and [32].

In [4], Branner and Hubbard conjectured that the assertion in theorem B is true for any polynomial.

Let f be a polynomial with real coefficients such that one real critical point has a bounded orbit and all other critical points escape to infinity. Then the Julia set J_f is a Cantor set if and only if the critical component of K_f is aperiodic. See [19] and [20]. In [9], Emerson gave a combinatorial condition for the Julia set of a polynomial to be a Cantor set and showed that there are polynomials fulfilling the condition.

The purpose of this paper is to give a proof of the above Branner-Hubbard's conjecture. We state the main result of this paper as the

Main Theorem. *Let f be a complex polynomial of degree $\geqslant 2$ and let* Crit *be the set of critical points of f with bounded orbits. Then the Julia set J_f of f is a Cantor set if and only if the critical component $K_f(c)$ is aperiodic for all $c \in$*Crit.

There are two important tools in our proof. One is a nested sequence of some critical pieces constructed by Kozlovski, Shen, and van Strien in [17] which we shall call "KSS nest". The other one is a covering lemma proved by Kahn and Lyubich recently, see [15]. This covering lemma has many important applications in complex dynamics, see [2] and [16].

The paper is organized as follows. In section 2, we present some definitions and reduce the Main Theorem to the Main Proposition. We summarize the construction of KSS nest in section 3. The proof of the Main Proposition is given in section 4. In section 5, we prove a stronger result than the Main Theorem which states that each wandering component of the filled-in Julia set for an arbitrary polynomial is a point.

2 Definitions and preliminary results

For a complex polynomial f of degree $d \geqslant 2$, it is well-known that the function

$$G : \mathbb{C} \to \mathbb{R}_+ \cup \{0\}$$

defined by

$$G(z) = \lim_{n\to\infty} \frac{1}{d^n} \log^+ |f^n(z)|$$

is continuous and satisfies

$$(1) \quad G(f(z)) = dG(z), \qquad (2) \quad K_f = \{z \in \mathbb{C} \mid G(z) = 0\},$$

see [3] and [7].

The *Branner-Hubbard puzzle* of f is constructed as follows. Choose a small number $r_0 > 0$ which is not a critical value of G such that the region $G^{-1}(0, r_0)$ contains no critical points of f. Then for each integer $k \geqslant 0$, the locus

$$G^{-1}([0, r_0 d^{-k})) = \{z \in \mathbb{C} | \, G(z) < r_0 d^{-k}\}$$

is the disjoint union of a finite number of open topological disks. Each such open disk will be called a puzzle piece P_k of depth k. Thus each point $x \in K_f$ determines a nested sequence $P_0(x) \supset P_1(x) \supset \cdots$ and $K_f(x) = \bigcap_{k\geqslant 0} P_k(x)$. By Grötzsch's inequality,

$$\mathrm{mod}\ (P_0(x) - K_f(x)) = \infty$$

if there exists a subsequence

$$P_{k_1'}(x) \supset P_{k_1}(x) \supset P_{k_2'}(x) \supset P_{k_2}(x) \supset \cdots$$

such that

$$\sum_{1=1}^{\infty} \mathrm{mod}\ (P_{k_i'}(x) - \overline{P_{k_i}(x)}) = \infty.$$

It follows that $K_f(x) = \bigcap_{k\geqslant 0} P_k(x) = \{x\}$, see [1] and [4].

The Julia set J_f is a Cantor set if and only if $K_f(x) = \{x\}$ for any $x \in K_f$.

If a component K of K_f contains critical points $c_1, c_2, \cdots, c_k$, then $P_n(c_1) = P_n(c_2) = \cdots = P_n(c_k)$ and

$$\deg(f|_{P_n(c_1)}) = (\deg_{c_1} f - 1) + (\deg_{c_2} f - 1) + \cdots + (\deg_{c_k} f - 1) + 1$$

for all $n \geqslant 0$. We can think of K as a component containing one critical point of degree $(\deg_{c_1} f - 1) + (\deg_{c_2} f - 1) + \cdots + (\deg_{c_k} f - 1) + 1$. We therefore assume each critical component of K_f contains only one critical point in the following. Take r_0 small enough such that each puzzle piece contains at most one critical point.

For each $x \in K_f$, the tableau $T(x)$ is defined in [4]. It is the two dimension array $P_{n,l}(x) = f^l(P_{n+l}(x))$. The position (n, l) is called critical if $P_{n,l}(x)$ contains a critical point of f. If $P_{n,l}(x)$ contains a critical point c, the position (n, l) is called a c-position. Let Crit be the set of critical points with bounded orbits. The tableau $T(c)$ of a critical point $c \in \text{Crit}$ is called periodic if there is a positive integer k such that $P_n(c) = f^k(P_{n+k}(c))$ for all $n \geq 0$. Otherwise, $T(c)$ is said to be aperiodic.

All the tableaus satisfy the following three rules

(T1) If $P_{n,l}(x) = P_n(c)$ for some critical point c, then $P_{i,l}(x) = P_i(c)$ for all $0 \leqslant i \leqslant n$.

(T2) If $P_{n,l}(x) = P_n(c)$ for some critical point c, then $P_{i,l+j}(x) = P_{i,j}(c)$ for $i + j \leqslant n$.

(T3) Let $T(c)$ be a tableau for some critical point c and $T(x)$ be any tableau. Assume

(a) $P_{n+1-l,l}(c) = P_{n+1-l}(c_1)$ for some critical point c_1 and $n > l \geqslant 0$, and $P_{n-i,i}(c)$ contains no critical points for $0 < i < l$.

(b) $P_{n,m}(x) = P_n(c)$ and $P_{n+1,m}(x) \neq P_{n+1}(c)$ for some $m > 0$.

Then $P_{n+1-l,m+l}(x) \neq P_{n+1-l}(c_1)$.

In order to show that the Julia set for a polynomial is a Cantor set, we shall use the polynomial-like mapping theory introduced by Douady and Hubbard in [8]. Recall that a *polynomial-like mapping* of degree d is a triple (U, V, g) where U and V are simply connected plane domains with $\overline{V} \subset U$, and $g : V \to U$ is a holomorphic proper mapping of degree d. The filled-in Julia set K_g of the polynomial-like mapping g is defined as

$$K_g = \{z \in V \mid g^n(z) \in U \text{ for all } n \geqslant 0\}.$$

Two polynomial-like mappings (U_1, V_1, g_1) and (U_2, V_2, g_2) of degree d are said to be *hybrid equivalent* if there exists a quasi-conformal homeomorphsim h from a neighborhood of K_{g_1} onto a neighborhood of K_{g_2}, conjugating g_1 and g_2 and such that $\bar{\partial} h = 0$ on K_{g_1}. The following theorem was proved by Douady and Hubbard in [8].

Theorem C(The straightening theorem). *(1) Every polynomial-like mapping (U, V, g) of degree d is hybrid equivalent to a polynomial of degree d.*

(2) If K_g is connected, then the polynomial is uniquely determined up to conjugation by an affine map.

If $T(c)$ is periodic of period k, then $(P_n(c), P_{n+k}(c), f^k)$ is a polynomial-like mapping of degree $\deg(f^k|_{K_f(c)}) \geqslant 2$ for some $n \geqslant 0$. The filled-in Julia

set of this polynomial-like mapping equals to $K_f(c)$. From the straightening theorem, $K_f(c)$ is quasi-conformally homeomorphic to the filled-in Julia set of a polynomial of degree $\deg(f^k|_{K_f(c)}) \geqslant 2$. The "only if" part in the Main Theorem is obvious. We always assume that each critical component of K_f is aperiodic before section 5. It is equivalent to assuming $T(c)$ is aperiodic for all $c \in \mathrm{Crit}$.

Definition 1. (1) The tableau $T(x)$ for $x \in K_f$ is *non-critical* if there exists an integer $n_0 \geqslant 0$ such that (n_0, j) is not critical for all $j > 0$.

(2) We say the forward orbit of x *combinatorially accumulates* to y, written as $x \to y$, if for any $n \geqslant 0$, there exists $j > 0$ such that $y \in P_{n,j}(x)$, i.e., $f^j(P_{n+j}(x)) = P_n(y)$. It is clear that if $x \to y$ and $y \to z$, then $x \to z$. For each critical point $c \in \mathrm{Crit}$, let

$$F(c) = \{c' \in \mathrm{Crit} \mid c \to c'\}$$

and

$$[c] = \{c' \in \mathrm{Crit} \mid c \to c' \text{ and } c' \to c\}.$$

(3) We say $P_{n+k}(c')$ is a *child* of $P_n(c)$ if $c' \in [c]$, $f^k(P_{n+k}(c')) = P_n(c)$, and $f^{k-1} : P_{n+k-1}(f(c')) \to P_n(c)$ is conformal.

(4) Suppose $c \to c$, i.e., $[c] \neq \emptyset$. We say $T(c)$ is *persistently recurrent* if $P_n(c_1)$ has only finitely many children for all $n \geqslant 0$ and all $c_1 \in [c]$. Otherwise, $T(c)$ is said to be *reluctantly recurrent*.

Take a small $r_0 > 0$ such that for any $c \in \mathrm{Crit}$, there are no c'-positions in the first row of $T(c)$ if $c \not\to c'$.

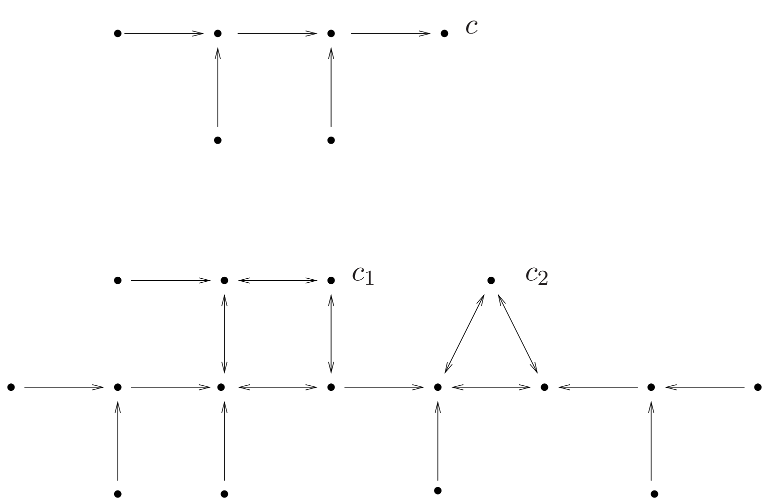

Figure 1

Let

$$\begin{aligned}
\mathrm{Crit_n} &= \{c \in \mathrm{Crit} \mid T(c) \text{ is non-critical}\},\\
\mathrm{Crit_p} &= \{c \in \mathrm{Crit} \mid T(c) \text{ is persistently recurrent}\},\\
\mathrm{Crit_r} &= \{c \in \mathrm{Crit} \mid T(c) \text{ is reluctantly recurrent}\},\\
\mathrm{Crit_{en}} &= \{c' \in \mathrm{Crit} \mid c' \not\to c' \text{ and } c' \to c \text{ for some } c \in \mathrm{Crit_n}\},\\
\mathrm{Crit_{ep}} &= \{c' \in \mathrm{Crit} \mid c' \not\to c' \text{ and } c' \to c \text{ for some } c \in \mathrm{Crit_p}\},\\
\mathrm{Crit_{er}} &= \{c' \in \mathrm{Crit} \mid c' \not\to c' \text{ and } c' \to c \text{ for some } c \in \mathrm{Crit_r}\}.
\end{aligned}$$

Then

$$\mathrm{Crit} = \mathrm{Crit_n} \cup \mathrm{Crit_p} \cup \mathrm{Crit_r} \cup \mathrm{Crit_{en}} \cup \mathrm{Crit_{ep}} \cup \mathrm{Crit_{er}}.$$

It is not a classification because these sets might intersect.

Consider the critical points c, c_1 and c_2 in Figure 1. The tableau $T(c)$ for c is non-critical. From Lemma 1 in the following, the tableau $T(c_1)$ for c_1 is reluctantly recurrent. The tableau $T(c_2)$ for c_2 is reluctantly recurrent or persistently recurrent.

Combined with arguments of Branner and Hubbard in [4], we have the following proposition.

Proposition 1. *(1) If $T(x)$ is non-critical, then $K_f(x) = \{x\}$.*

(2) Suppose $c \in \mathrm{Crit_n} \cup \mathrm{Crit_r}$. Then $K_f(c) = \{c\}$ and $K_f(x) = \{x\}$ for any $x \to c$.

Proof. Let $\mathcal{P}_n$ be the collection of all puzzle pieces of depth n. It has only finitely many pieces. Hence

$$\nu_n = \min\{\mathrm{mod}(P_n - \overline{P_{n+1}}) \mid P_n \in \mathcal{P}_n,\ P_{n+1} \in \mathcal{P}_{n+1}, \text{with } P_{n+1} \subset P_n\} > 0.$$

(1) Since $T(x)$ is non-critical, there exists an integer $n_0 \geqslant 0$ such that (n_0, j) is not a critical position for all $j > 0$. For any $k \geqslant 1$, $\deg(f^k|_{P_{n_0+k}(x)}) \leqslant \deg(f|_{P_{n_0+1}(x)})$ and

$$\begin{aligned}
\mathrm{mod}(P_{n_0+k}(x) - \overline{P_{n_0+k+1}(x)}) &\geqslant \frac{1}{\deg(f^k|_{P_{n_0+k}(x)})}\mathrm{mod}(P_{n_0}(f^k(x)) - \overline{P_{n_0+1}(f^k(x))})\\
&\geqslant \frac{\nu_{n_0}}{\deg(f|_{P_{n_0+1}(x)})} > 0.
\end{aligned} \tag{2.1}$$

This yields

$$\sum_{n=1}^{\infty} \mathrm{mod}\ (P_n(x) - \overline{P_{n+1}(x)}) = \infty.$$

Hence, $K_f(x) = \{x\}$.

(2) If $c \in \mathrm{Crit_n}$, then $T(c)$ is non-critical and $K_f(c) = \{c\}$. There exists an integer $n_0 \geq 0$ such that

$$\deg(f^k|_{P_{n_0+k}(c)}) = \deg_c f$$

for all $k \geqslant 1$.

For any $x \to c$, let l_k be the first moment such that $f^{l_k}(x) \in P_{n_0+k}(c)$, *i.e.* (n_0+k, l_k) is the first c-position on the (n_0+k)-th row in $T(x)$. By tableau rules (T1) and (T2), there is at most one c'-position on the diagonal

$$\{(n,m) \mid n+m = n_0+k+l_k, \quad n_0+k < n \leqslant n_0+k+l_k\}$$

for any $c' \in \mathrm{Crit} - \{c\}$. Therefore, $f^{l_k+k}(P_{n_0+k+l_k}(x)) = P_{n_0}(f^{l_k+k}(x))$ and $\deg(f^{l_k+k}|_{P_{n_0+k+l_k}(x)}) \leqslant D_1 < \infty$ for any $k \geq 1$, where D_1 is an integer independent of k. We have

$$\mathrm{mod}\ (P_{n_0+k+l_k}(x) - \overline{P_{n_0+k+l_k+1}(x)}) \geqslant \frac{\nu_{n_0}}{D_1} > 0.$$

So $K_f(x) = \{x\}$.

If $c \in \mathrm{Crit_r}$, then there exist an integer $n_0 \geq 0$, $c' \in [c]$, $c_1 \in [c]$ and infinitely many integers $k_n \geqslant 1$ such that $\{P_{n_0+k_n}(c')\}_{n \geqslant 1}$ are children of $P_{n_0}(c_1)$. Let m_n be the first moment such that $f^{m_n}(c) \in P_{n_0+k_n}(c')$. There is at most one $\widetilde{c}$-position on the diagonal

$$\{(n,m) \mid n+m = n_0+k_n+m_n, \quad n_0+k_n < n \leqslant n_0+k_n+m_n\}$$

in $T(c)$ for any $\widetilde{c} \in \mathrm{Crit} - \{c\}$. Therefore, $f^{m_n+k_n}(P_{n_0+k_n+m_n}(c)) = P_{n_0}(c_1)$ and $\deg(f^{m_n+k_n}|_{P_{n_0+k_n+m_n}(c)}) \leqslant D_2 < \infty$ for any $n \geq 1$, where D_2 is an integer independent of n. We have

$$\mathrm{mod}\ (P_{n_0+k_n+m_n}(c) - \overline{P_{n_0+k_n+m_n+1}(c)}) \geqslant \frac{\nu_{n_0}}{D_2} > 0$$

and $K_f(c) = \{c\}$.

Suppose $x \to c$ for some $c \in \mathrm{Crit_r}$. Let l_n be the first moment such that $f^{l_n}(x) \in P_{n_0+k_n+m_n}(c)$ and let $t_n = k_n + m_n + l_n$. By the same method, we have $f^{t_n}(P_{n_0+t_n}(x)) = P_{n_0}(c_1)$ and $\deg(f^{t_n}|_{P_{n_0+t_n}(x)}) \leqslant D_3 < \infty$ for any $n \geq 1$, where D_3 is an integer independent of n. Hence

$$\mathrm{mod}\ (P_{n_0+t_n}(x) - \overline{P_{n_0+t_n+1}(x)}) \geqslant \frac{\nu_{n_0}}{D_3} > 0$$

and $K_f(x) = \{x\}$. □

From Proposition 1 and $\mathrm{Crit} = \mathrm{Crit_n} \cup \mathrm{Crit_p} \cup \mathrm{Crit_r} \cup \mathrm{Crit_{en}} \cup \mathrm{Crit_{ep}} \cup \mathrm{Crit_{er}}$, we can reduce the Main Theorem to the following proposition.

Main Proposition. *If $c \in \mathrm{Crit_p}$, then $K_f(c) = \{c\}$ and $K_f(x) = \{x\}$ for all $x \to c$.*

The following lemma will be used in sections 3 and 4.

Lemma 1. *If $T(c)$ is persistently recurrent, then $F(c) = [c]$.*

Proof. Suppose $c \to c'$ and $c' \not\to c$. If there exists a column where each position is c'-position, then $c' \to c$. It contradicts with our assumption. Hence there are infinitely many c'-positions $\{(n_k, m_k)\}_{k \geqslant 1}$ in $T(c)$ such that $(n_k + 1, m_k)$ is not critical and $\lim_{k \to \infty} n_k = \infty$. By the tableau rule (T2) and the choice of r_0, there are no $\tilde{c}$-positions on the diagonal

$$\{(n, m) \mid n + m = n_k + m_k, \quad 0 \leqslant n \leqslant n_k\}$$

for any $\tilde{c} \in [c]$.

Let $(0, t_k)$ be a $c_2(k)$-position on the right of $(0, n_k + m_k)$ for some $c_2(k) \in [c]$ such that there are no $\tilde{c}$-positions between $(0, n_k + m_k)$ and $(0, t_k)$ for any $\tilde{c} \in [c]$. Then there are no c''-positions on the diagonal

$$\{(n, m) \mid n + m = t_k, \quad 0 < n < t_k - m_k\}$$

for any $c'' \in F(c)$. Hence all positions on this diagonal are not critical. Let s_k be the largest integer between 0 and m_k such that $(t_k - s_k, s_k)$ is a critical position. Say it is a $c_1(k)$-position for some $c_1(k) \in [c]$. See Figure 2. Take a subsequence $\{k_j\}$ such that $c_2(k_j) = c_2$ for some $c_2 \in [c]$. Then the critical piece $P_0(c_2)$ has infinitely many children. This is impossible because $T(c)$ is persistently recurrent. $\square$

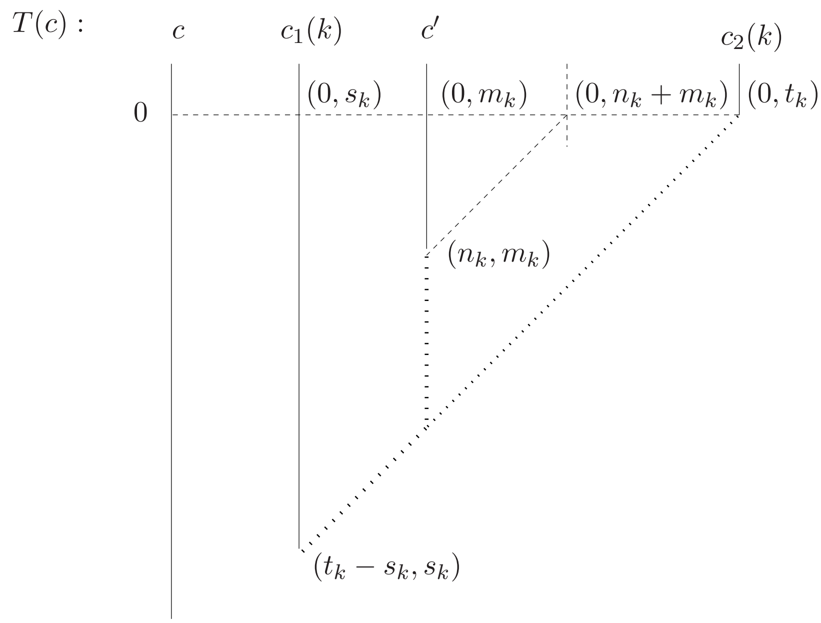

Figure 2

The proof of the Main Proposition will be given in section 4.

3 KSS nest

For completeness, we summarize the construction of a critical nest and related results given by Kozlovski, Shen, and van Strien in [17]. Such nest will be called *KSS nest*. Principal nest and modified principal nest are used to study the dynamics of unicritical polynomials, see [2], [5], [16], [21] and [22]. In [22], Lyubich proved the linear growth of its "principal moduli" for quadratic polynomials. This yields the density of hyperbolic maps in the real quadratic family. The same result is also obtained by Graceyk and Świątek in [11]. See also [24] and [35]. Recently, the local connectivity of Julia sets and combinatorial rigidity for unicritical polynomials are proved in [16] and [2] by means of principal nest and modified principal nest. For dynamics of multimodal maps, see [33] and [34].

Let A be an open set and $x \in A$. We denote the connected component of A containing x by $\mathrm{Comp}_x(A)$. Given an open set X consisting of finitely many puzzle pieces(not necessarily the same depth) such that $f^n(z) \notin \overline{X}$ for any $z \in \partial X$ and any $n \geqslant 1$, let

$$D(X) = \{z \in \mathbb{C} \mid \exists k \geqslant 1 \ s.t. \ f^k(z) \in X\}.$$

The first entry map

$$R_X : D(X) \to X$$

is defined by $z \mapsto f^{k(z)}(z)$, where $k(z) \geqslant 1$ is the smallest integer with $f^{k(z)}(z) \in X$. Let I be a component of $D(X)$. Then there exists an integer k such that $k(z) = k$ for any $z \in I$ and $f^k(I)$ is a connected component of X. The orbit

$$\{I, \, f(I), \, \cdots \, , \, f^{k-1}(I)\}$$

meets each critical point at most once and the degree of f^k on I is uniformly bounded. For any $z \in D(X)$, let $L_z(X)$ be the connected component of $D(X)$ containing z. We further define $\hat{L}_z(X) = \mathrm{Comp}_z(X)$ for $z \in X$ and $\hat{L}_z(X) = L_z(X)$ for $z \in D(X) - X$.

Suppose $T(c_0)$ is persistently recurrent, then $F(c_0) = [c_0]$. Let

$$b = \#[c_0], \ d_0 = \deg_{c_0} f, \ d_{max} = \max\{\deg_c f \,|\, c \in [c_0]\}$$

and

$$\mathrm{orb}([c_0]) = \cup_{n \geqslant 0} f^n([c_0]).$$

For any puzzle piece I containing c_0, we construct puzzle pieces $P'_c \subset\subset P_c$ for any $c \in [c_0]$ as follows. Let $T_0 = I$ and $J_0 = L_{c_0}(I)$. If $R_I(c') \in J_0$ for any $c' \in [c_0] - \{c_0\}$, we take $P_c = \hat{L}_c(T_0)$ and $P'_c = \hat{L}_c(J_0)$ for any $c \in [c_0]$. If $R_I(c_1) \notin J_0$ for some $c_1 \in [c_0] - \{c_0\}$, let $T_1 = J_0 \cup \mathrm{Comp}_{c_1}(R_I^{-1}(L_{R_I(c_1)}(I)))$ and $J_1 = L_{c_0}(T_1) \cup L_{c_1}(T_1)$. If $R_{T_1}(c') \in J_1$ for any $c' \in [c_0] - \{c_0, c_1\}$, we take $P_c = \hat{L}_c(T_1)$ and $P'_c = \hat{L}_c(J_1)$ for any $c \in [c_0]$. Repeating this process, we have $T_m = J_{m-1} \cup \mathrm{Comp}_{c_m}(R_{T_{m-1}}^{-1}(L_{R_{T_{m-1}}(c_m)}(T_{m-1})))$ and

$J_m = \bigcup_{0\leqslant i\leqslant m} L_{c_i}(T_m)$ for some $m < b$ such that $R_{T_m}(c') \in J_m$ for any $c' \in [c_0] - \{c_0, c_1, ..., c_m\}$. Let $P_c = \hat{L}_c(T_m)$ and $P'_c = \hat{L}_c(J_m)$ for any $c \in [c_0]$. These two pieces $P'_c \subset\subset P_c$ satisfy the following two properties

(P1) There exists an integer l_c such that $f^{l_c}(P_c) = I$, $\deg(f^{l_c}: P_c \to I) \leqslant {d_{max}}^{b^2-b}$ and $\#\{i|\, c_0 \in f^i(P_c),\, 0 \leqslant i < l_c\} \leqslant b-1$. The piece P'_c is also a pull-back of I.

(P2) For each $x \in (P_c - P'_c) \cap \mathrm{orb}([c_0])$, there exist a positive integer k, a puzzle piece $V(x)$ containing x and $\tilde{c} \in [c_0]$ such that $f^k : V(x) \to P_{\tilde{c}}$ is conformal. In fact, let $k \geqslant 1$ be the first moment such that $f^k(x) \in \cup_{\tilde{c}\in[c_0]} P_{\tilde{c}}$, $f^k(x) \in P_{\tilde{c}}$ for some $\tilde{c} \in [c_0]$, and let $V(x)$ be the component of $f^{-k}(P_{\tilde{c}})$ containing x, then $V(x) \subset P_c - P'_c$ and $f^k : V(x) \to P_{\tilde{c}}$ is conformal.

Since $T(c_0)$ is persistently recurrent, each P_c has only finitely many children. Let Q_c be the last child of P_c. Then there exists an integer $v_c \geqslant 1$, largest among all the children of P_c, such that $f^{v_c}(Q_c) = P_c$. The set Q_c contains a critical point $c' \in [c_0]$. Let

$$v = \max\{v_c|\, c \in [c_0]\}.$$

Suppose $v = v_{c_1}$ for some $c_1 \in [c_0]$. By (P2) as above and the maximality of v, we have $f^v(c') \in P'_{c_1}$ and

$$(Q_{c_1} - Q'_{c_1}) \cap \mathrm{orb}([c_0]) = \emptyset,$$

where Q'_{c_1} is the connected component of $f^{-v}(P'_{c_1})$ containing c'.

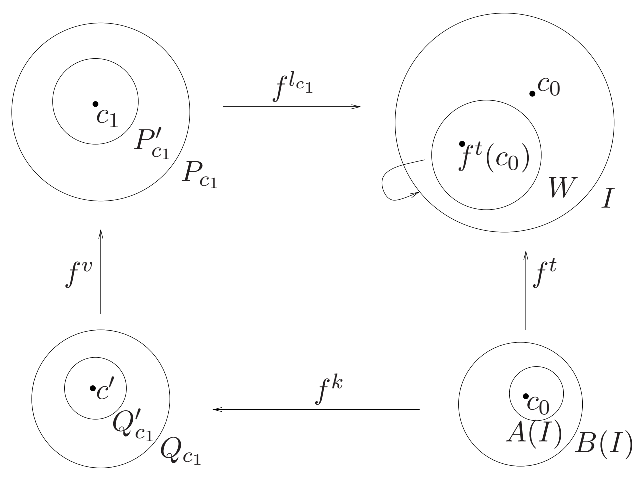

Figure 3

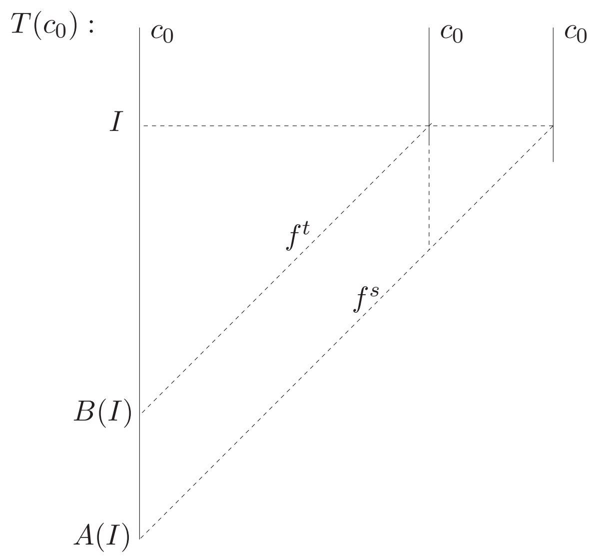

Figure 4

Let $B(I) = \hat{L}_{c_0}(Q_{c_1})$, $f^k(B(I)) = Q_{c_1}$, $t = k+v+l_{c_1}$, and $W = L_{f^t(c_0)}(I)$. Then $f^{l_{c_1}}(P'_{c_1}) \subset W$ because P'_{c_1} is mapped to I and

$$f^t(c_0) = f^{l_{c_1}}(f^{k+v}(c_0)) \in f^{l_{c_1}}(P'_{c_1}) \cap W \neq \emptyset.$$

Let $A(I)$ be the connected component of $f^{-t}(W)$ containing c_0 and $f^s(A(I)) = I$. See Figures 3 and 4.

Definition 2. Given a puzzle piece P containing c_0, a *successor* of P is a piece of the form $\hat{L}_{c_0}(Q)$, where Q is a child of $\hat{L}_c(P)$ for some $c \in [c_0]$. See Figure 5.

It is clear that $L_{c_0}(P)$ is a successor of P. Since $T(c_0)$ is aperiodic and is persistently recurrent, P has at least two successors and has only finitely many successors. Let $\Gamma(P)$ be the *last successor* of P. Then there exists an integer $q \geqslant 1$, largest among all of the successors of P, such that $f^q(\Gamma(P)) = P$.

We state some facts which will be used in the following as

(F1) $f^t(B(I)) = I$, $\deg(f^t|_{B(I)}) \leqslant {d_{max}}^{b^2}$ and $\#\{i|\, c_0 \in f^i(B(I)), 0 \leqslant i < t\} \leqslant b$,

(F2) $f^s(A(I)) = I$, $\deg(f^s|_{A(I)}) \leqslant {d_{max}}^{b^2+b}$ and $\#\{i|\, c_0 \in f^i(A(I)), 0 \leqslant i < s\} \leqslant b+1$,

(F3) $(B(I) - A(I)) \cap \mathrm{orb}([c_0]) = \emptyset$,

(F4) $f^q(\Gamma(P)) = P$ and $\deg(f^q|_{\Gamma(P)}) \leqslant {d_{max}}^{2b-1}$,

(F5) $f^i(\Gamma(P))$ does not contain c_0 for all $0 < i < q$.

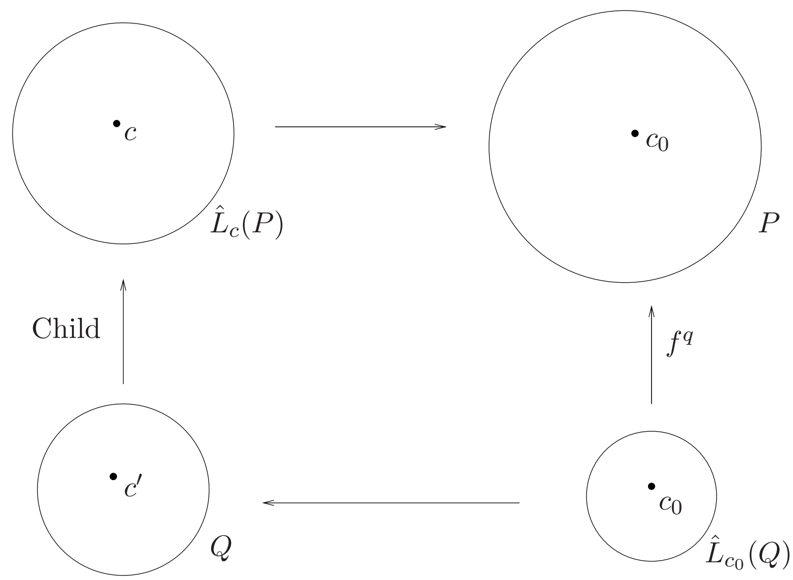

Figure 5. A successor of P

Now we can define the *KSS nest* in the following way: I_0 is a given piece containing c_0 and for $n \geqslant 0$,

$$\begin{aligned}
&L_n = A(I_n),\\
&M_{n,0} = K_n = B(L_n),\\
&M_{n,j+1} = \Gamma(M_{n,j}) \text{ for } 0 \leqslant j \leqslant T-1,\\
&I_{n+1} = M_{n,T} = \Gamma^T(K_n) = \Gamma^T(B(A(I_n))),
\end{aligned}$$

with $T = 3b$.

Suppose $f^{s_n}(L_n) = I_n$, $f^{t_n}(K_n) = L_n$, $f^{q_{n,j}}(M_{n,j}) = M_{n,j-1}$ for $1 \leqslant j \leqslant T$, and $q_n = \sum_{j=1}^{T} q_{n,j}$. See Figure 6.

Let $p_n = q_{n-1} + s_n + t_n$. Then $f^{p_n}(K_n) = K_{n-1}$. From (F1), (F2), and (F4), we have

$$d_0^{3b+2} \leqslant \deg(f^{p_n}|_{K_n}) \leqslant d_1,$$

where $d_1 = d_{max}^{8b^2-2b}$.

For any puzzle piece J containing c_0, let

$$r(J) = \min\{k(z) \mid z \in D(J) \cap J\},$$

where $k(z)$ is the smallest positive integer such that $f^{k(z)}(z) \in J$. It is easy to prove that

(1) $r(J_1) \geqslant r(J_2)$ if $J_1 \subset J_2$.

(2) $r(J) \geqslant k$ if $c_0 \in J \subset J'$, $f^k : J \to J'$ and $c_0 \notin f^i(J)$ for $0 < i < k$.

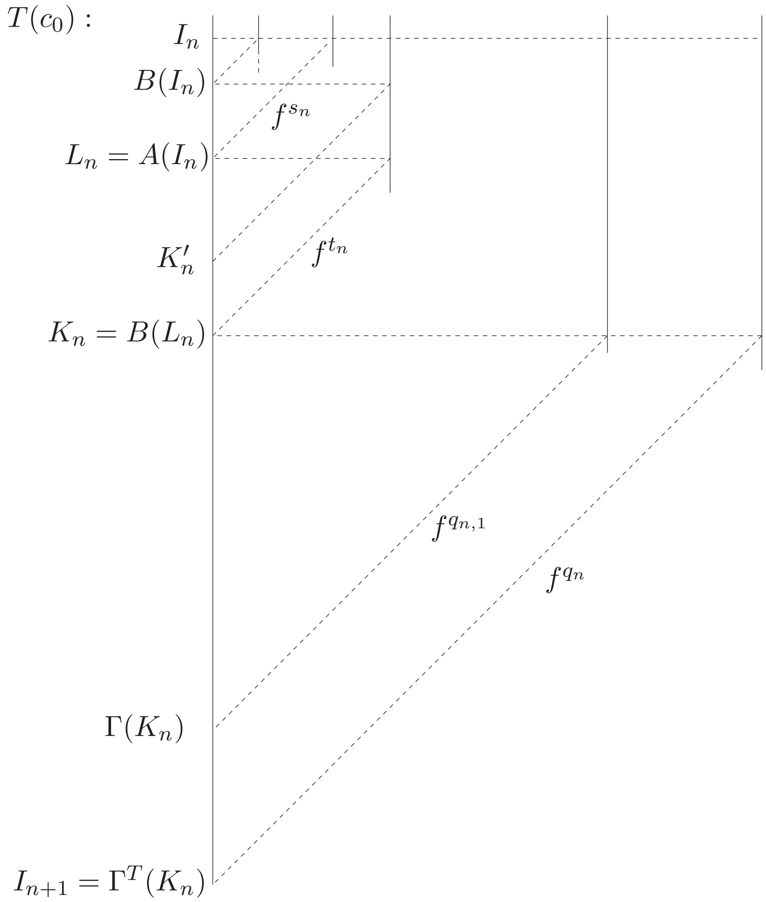

Figure 6. KSS nest

The following lemma is a slight modification of Lemma 8.2 in [17], and the proof is very much the same.

Lemma 2. *Let* $T = 3b$. *Then*

(1) $r(I_n) \leqslant s_n \leqslant (b+1)r(L_n)$;

(2) $r(L_n) \leqslant t_n \leqslant br(K_n)$;

(3) $2r(M_{n,j-1}) \leqslant q_{n,j} \leqslant r(M_{n,j})$ *for* $1 \leqslant j \leqslant T$;

(4) $p_{n+1} \geqslant 2p_n$, $\dfrac{p_n}{t_n} \leqslant b+4$.

Proof. (1) The inequality $r(L_n) \leqslant s_n$ is obvious. Let $G_j = f^j(L_n)$ and $0 = j_0 < j_1 < \cdots < j_v = s_n$ be all the integers such that $c_0 \in G_{j_i}$. Then $v \leqslant b+1$ and $f^{j_{i+1}-j_i} : G_{j_i} \to G_{j_{i+1}}$. Note that $c_0 \notin G_k$ for $j_i < k < j_{i+1}$.

Hence $j_{i+1} - j_i \leqslant r(G_{j_i}) \leqslant r(G_0) = r(L_n)$ and

$$s_n = \sum_{i=0}^{v-1}(j_{i+1} - j_i) \leqslant vr(L_n) \leqslant (b+1)r(L_n).$$

(2) The proof is similar to that of (1).

(3) Let $1 \leqslant j \leqslant T$. Since $M_{n,j}$ is a successor of $M_{n,j-1}$ with $f^{q_{n,j}} : M_{n,j} \to M_{n,j-1}$ for some $q_{n,j}$ and $c_0 \notin f^i(M_{n,j})$ for $0 < i < q_{n,j}$, we have $q_{n,j} \leqslant r(M_{n,j})$. Let k be the smallest positive integer with $f^k(c_0) \in M_{n,j-1}$ and $J = L_{c_0}(M_{n,j-1})$. Then $f^k(J) = M_{n,j-1}$ and J is the first successor. Because $M_{n,j-1}$ has at least two successors and $M_{n,j}$ is the last one, we have $q_{n,j} - k > 0$. Denote $x = f^k(c_0)$. Then $x \in M_{n,j-1} \cap D(M_{n,j-1})$ and $f^{q_{n,j}-k}(x) \in M_{n,j-1}$. It follows that $q_{n,j} = (q_{n,j} - k) + k \geqslant r(M_{n,j-1}) + r(M_{n,j-1}) = 2r(M_{n,j-1})$.

(4) By (3),

$$2^j r(K_n) = 2^j r(M_{n,0}) \leqslant q_{n,j} \leqslant \frac{1}{2^{T-j}} r(M_{n,T}) = \frac{1}{2^{T-j}} r(I_{n+1})$$

for any $n \geqslant 1$ and $1 \leqslant j \leqslant T$. From (1) and (2), we have

$$\begin{aligned}
p_{n+1} &= q_n + s_{n+1} + t_{n+1} \\
&= \sum_{j=1}^{T} q_{n,j} + s_{n+1} + t_{n+1} \\
&\geqslant (2^{T+1} - 2)r(K_n) + r(I_{n+1}) + r(L_{n+1}) \\
&\geqslant 2^{T+1} r(K_n) = 2^{3b+1} r(K_n)
\end{aligned}$$

and

$$\begin{aligned}
p_n &= q_{n-1} + s_n + t_n \\
&= \sum_{j=1}^{T} q_{n-1,j} + s_n + t_n \\
&< 2r(I_n) + (b+1)r(L_n) + br(K_n) \\
&\leqslant (2b+3)r(K_n).
\end{aligned}$$

Therefore, $p_{n+1} \geqslant 2p_n$.

The second inequality can be obtained from the following fact

$$\begin{aligned}
p_n &= q_{n-1} + s_n + t_n \\
&< 2r(I_n) + (b+1)r(L_n) + t_n \\
&\leqslant (b+3)r(L_n) + t_n \\
&\leqslant (b+4)t_n.
\end{aligned}$$

□

4 Proof of the Main Proposition

Let $K_n' = \mathrm{Comp}_{c_0} f^{-t_n}(B(I_n))$. The conditions $F(c_0) = [c_0]$ and $(B(I_n) - L_n) \cap \mathrm{orb}([c_0]) = \emptyset$ imply

$$d_0 \leqslant \deg(f^{t_n}|_{K_n'}) = \deg(f^{t_n}|_{K_n}) \leqslant d_1.$$

Let $\mu_n = \bmod\ (K_n' - \overline{K_n})$.

The main result in this section is the following lemma.

Lemma 3. $\liminf_{n\to\infty} \mu_n > 0$.

We first state a covering lemma recently given by Kahn and Lyubich which will play a crucial rule in the proof of Lemma 3.

Kahn-Lyubich Lemma ([15]). *Fix some $\eta > 0$. Let $A \subset\subset A' \subset \mathrm{int}(U)$ and $B \subset\subset B' \subset \mathrm{int}(V)$ be two nests of Jordan disks. Let $f : (U, A', A) \to (V, B', B)$ be a holomorphic proper mapping between the respective disks, and let $D = \deg(f|_U)$ and $d = \deg(f|_{A'})$. Assume the following collar property*

$$\bmod\ (B' - \overline{B}) \geqslant \eta\ \bmod\ (U - \overline{A}).$$

Then there exists an $\epsilon > 0$ (depending on η and D) such that

$$\bmod\ (V - \overline{B}) \leqslant C\eta^{-1}d^2\ \bmod\ (U - \overline{A})$$

or

$$\bmod\ (U - \overline{A}) \geqslant \epsilon,$$

where C is an absolute constant.

Proof of Lemma 3

Suppose $\liminf_{n\to\infty} \mu_n = 0$. Let $\mu_{k_n} = \min\{\mu_1, \mu_2, \ldots, \mu_n\}$. Then $\lim_{n\to\infty} k_n = \infty$ and $\lim_{n\to\infty} \mu_{k_n} = 0$. Take an integer j_0 satisfying

$$2^{3b(j_0-1)} \geqslant (b+1)(2b+9)$$

and a large integer N. Let $M = p_{k_n-j_0} + p_{k_n-j_0-1} + \cdots + p_{k_n-N+1}$. Then

(1) $M < 2p_{k_n-j_0}$ (by Lemma 2(4)),

(2) $f^M(K_{k_n-j_0}) = K_{k_n-N}$,

(3) $d_0^{(3b+2)(N-j_0)} \leqslant D = \deg(f^M|_{K_{k_n-j_0}}) \leqslant {d_1}^{N-j_0}$,

where $d_0 = \deg_{c_0} f$ and d_1 is the constant in section 3 depending only on b and d_{max}.

For any $x \in K_{k_n-j_0} \cap \mathrm{orb}([c_0])$, let $y = f^M(x)$, $B_y = \hat{L}_y(K_{k_n-j_0})$, $f^l(B_y) = K_{k_n-j_0}$ and $B_y' = \hat{L}_y(K_{k_n-j_0}')$. See Figure 7.

Let $A_x = \mathrm{Comp}_x f^{-M}(B_y)$ and $A_x' = \mathrm{Comp}_x f^{-M}(B_y')$. From the conditions $(K_{k_n-j_0}' - K_{k_n-j_0}) \cap \mathrm{orb}([c_0]) = \emptyset$ and (T3), we have

$$\deg(f^M|_{A_x'}) = \deg(f^M|_{A_x}).$$

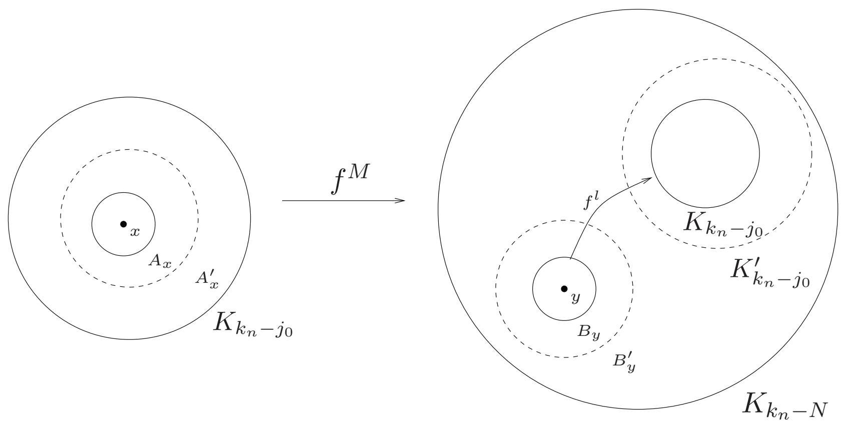

Figure 7

Claim 1. *For any* $x \in K_{k_n-j_0} \cap \mathrm{orb}([c_0])$,

$$\deg(f^M|_{A'_x}) = \deg(f^M|_{A_x}) \leqslant d_2,$$

where d_2 *is a constant depending only on* b *and* d_{max}*.*

Proof. Let $K_{k_n-j_0} = P_{n_0}(c_0)$. Suppose (n_0, m_1) is the first c_0-position on the right of $(n_0, t_{k_n-j_0})$ in $T(x)$. Let $W_1 = P_{n_0+m_1-t_{k_n-j_0}, t_{k_n-j_0}}(x)$ and $l_1 = m_1 - t_{k_n-j_0}$. Then $f^{l_1}(W_1) = P_{n_0}(c_0)$ and $\deg(f^{l_1}|_{W_1}) \leqslant d^b_{max}$. Repeating this process, we have infinitely many c_0-positions $\{(n_0, m_i)\}_{i\geqslant 1}$ such that (n_0, m_i) is the first c_0-position on the right of $(n_0, m_{i-1} + t_{k_n-j_0})$ for each $i \geqslant 1$ in $T(x)$. For any $i \geqslant 1$, $f^{l_i}(W_i) = P_{n_0}(c_0)$ and $\deg(f^{l_i}|_{W_i}) \leqslant d^b_{max}$, where $W_i = P_{n_0+l_i, m_{i-1}+t_{k_n-j_0}}(x)$, $l_i = m_i - m_{i-1} - t_{k_n-j_0}$. Let $L \geqslant 1$ be the smallest integer such that $m_L \geqslant M + l$. Then $(L-1)t_{k_n-j_0} \leqslant m_{L-1} < M$. By Lemma 2(4) and (F1),

$$L - 1 \leqslant \frac{M}{t_{k_n-j_0}} < \frac{2p_{k_n-j_0}}{t_{k_n-j_0}} \leqslant 2(b+4) = 2b+8$$

and

$$\deg(f^M|_{A_x}) \leqslant \Big(\deg(f^{t_{k_n-j_0}}|_{K_{k_n-j_0}})\Big)^L \cdot \prod_{j=1}^{L} (\deg f^{l_i}|_{W_i}) \leqslant d_2,$$

where $d_2 = d_{max}^{(b^2+b)(2b+9)}$. See Figure 8. ∎

Suppose $f^{M'}(I_{k_n}) = K_{k_n-j_0}$ and $f^{\sigma}(B(I_{k_n})) = I_{k_n}$, where

$$M' = q_{k_n-1} + p_{k_n-1} + \cdots + p_{k_n-j_0+1}.$$

From (F1) and $\deg(f^{p_n}|_{K_n}) \leqslant d_1$, we have $\deg(f^{\sigma}|_{B(I_{k_n})} \leqslant d^{b^2}_{max}$ and

$$\deg(f^{M'}|_{I_{k_n}}) \leqslant d^{b^2}_{max} d_1^{j_0-1},$$

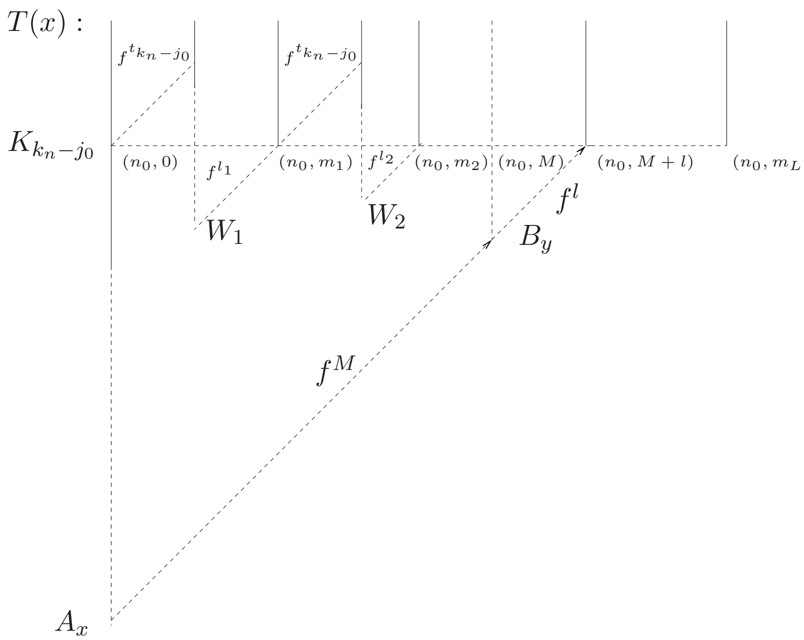

Figure 8

where $d_1 = d_{max}^{8b^2-2b}$ is obtained in section 3.

Let $x = f^{M'+\sigma}(c_0)$ and let A_x be the puzzle piece constructed as above. See Figure 9.

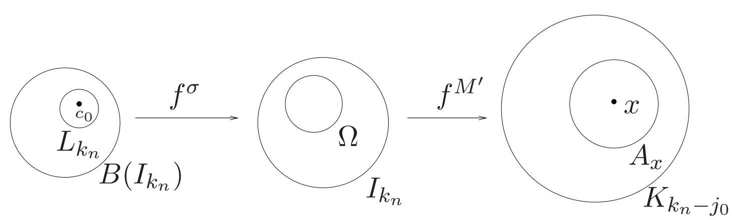

Figure 9

Claim 2. *Let $\Omega = f^\sigma(L_{k_n}) \subset I_{k_n}$. Then $f^{M'}(\Omega) \subset A_x$.*

Proof. Suppose $f^r(\Omega) = I_{k_n}$, then $f^{r+M'}(\Omega) = K_{k_n-j_0}$ and $r \geqslant r(I_{k_n})$. See Figure 10.

Let $K_{k_n-j_0} = P_{n_0}(c_0)$. Suppose $(n_0, v_0), (n_0, v_1), \ldots, (n_0, v_k)$ are all c_0-positions between $(n_0, M'+\sigma)$ and $(n_0, M'+\sigma+r)$ in $T(c_0)$ with $v_0 = M'+\sigma$ and $v_k = M'+\sigma+r$. See Figure 11.

Subclaim. *For all $0 \leqslant i \leqslant k-1$, $v_{i+1} - v_i \leqslant q_{k_n-j_0,1}$.*

Proof. We recall that $q_{k_n-j_0,1} \geqslant 1$ is the integer, largest among all of the successors of $P_{n_0}(c_0)$, such that $f^{q_{k_n-j_0,1}}(\Gamma(P_{n_0}(c_0))) = P_{n_0}(c_0)$.

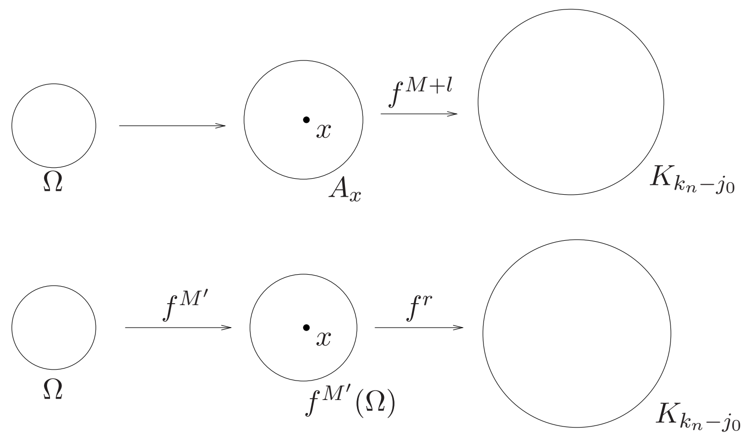

Figure 10

If $(n_0+v_{i+1}-v_i, v_i)$ is critical, then $P_{n_0+v_{i+1}-v_i}(c_0)$ is a successor of $P_{n_0}(c_0)$ and $v_{i+1}-v_i \leqslant q_{k_n-j_0,1}$.

Suppose that $(n_0+v_{i+1}-v_i, v_i)$ is not critical. Let k_i be the smallest integer between $v_{i+1}-v_i$ and v_{i+1} such that $(n_0+k_i, v_{i+1}-k_i)$ is a critical position, say it is a c-position, see Figure 11. Then $c \in [c_0]$. If $c = c_0$, then $P_{n_0+k_i}(c_0)$ is a successor of $P_{n_0}(c_0)$ and $v_{i+1}-v_i < k_i \leqslant q_{k_n-j_0,1}$. If $c \neq c_0$, let $P_{n_0+l_i}(c_0) = L_{c_0}(P_{n_0+k_i}(c))$. Then $P_{n_0+l_i}(c_0)$ is a successor of $P_{n_0}(c_0)$ and $v_{i+1}-v_i < k_i < l_i \leqslant q_{k_n-j_0,1}$.

□

By the Subclaim and Lemma 2(3),

$$\#\{i|\, f^i(f^{M'}(\Omega)) \subset K_{k_n-j_0}, 0 \leqslant i < r\} \geqslant \frac{r}{q_{k_n-j_0,1}} \geqslant \frac{r(I_{k_n})}{r(I_{k_n-j_0+1})} \geqslant 2^{3b(j_0-1)},$$

since $r(I_{n+1}) \geqslant 2^{3b} r(K_n) \geqslant 2^{3b} r(I_n)$ for all $n \geqslant 0$. See Figure 11.

By $K_{k_n-j_0} = B(L_{k_n-j_0})$ and (F1), we have

$$\#\{i|\, c_0 \in f^i(K_{k_n-j_0}), 0 \leqslant i \leqslant t_{k_n-j_0}\} \leqslant b+1.$$

For each $1 \leqslant j \leqslant L$, (n_0, k) is not c_0-position for $m_j - l_j < k < m_j$. Hence

$$\begin{aligned}\#\{i|\, f^i(A_x) \subset K_{k_n-j_0}, 0 \leqslant i < M+l\} &\leqslant L \cdot \#\{i|\, c_0 \in f^i(K_{k_n-j_0}), 0 \leqslant i \leqslant t_{k_n-j_0}\} \\ &\leqslant (b+1)L < (b+1)(2b+9).\end{aligned}$$

The integers L, m_j, and l_j are the same as in the proof of Claim 1. See Figure 8.

The condition $2^{3b(j_0-1)} \geqslant (b+1)(2b+9)$ implies that

$$f^{M'}(\Omega) \subset A_x.$$

□

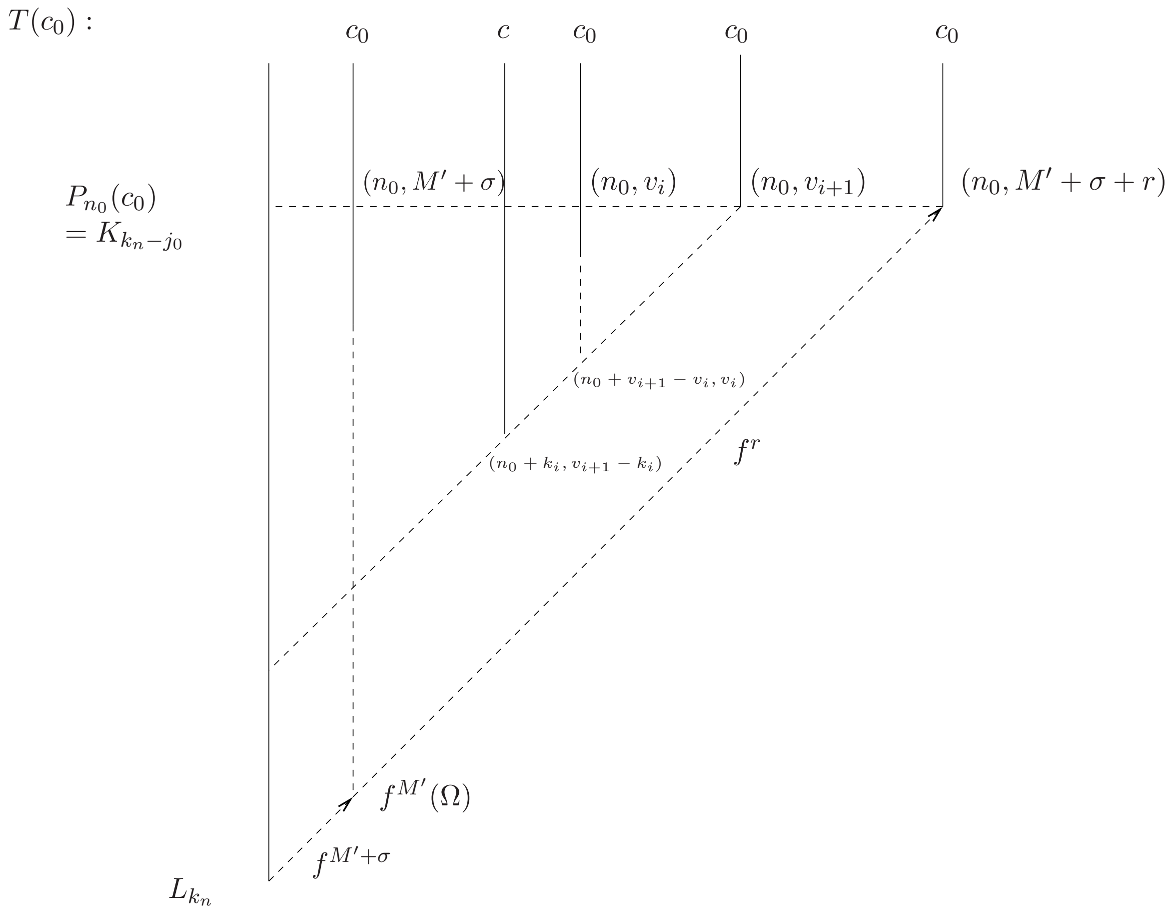

Figure 11

Claim 3. *There exists a positive constant* η *depending only on* b *and* d_{max} *such that*

$$\text{mod }(B'_y - \overline{B_y}) \geqslant \eta \text{ mod }(K_{k_n - j_0} - \overline{A_x}).$$

Proof. Since $\deg(f^l|_{B_y}) = \deg(f^l|_{B'_y}) \leqslant d_{max}^{b-1}$, we have

$$\begin{aligned} \text{mod }(B'_y - \overline{B_y}) &= \frac{1}{\deg(f^l|_{B_y})} \text{ mod }(K'_{k_n - j_0} - \overline{K_{k_n - j_0}}) \\ &\geqslant {d_{max}}^{-(b-1)} \mu_{k_n - j_0} \\ &\geqslant {d_{max}}^{-(b-1)} \mu_{k_n}. \end{aligned}$$

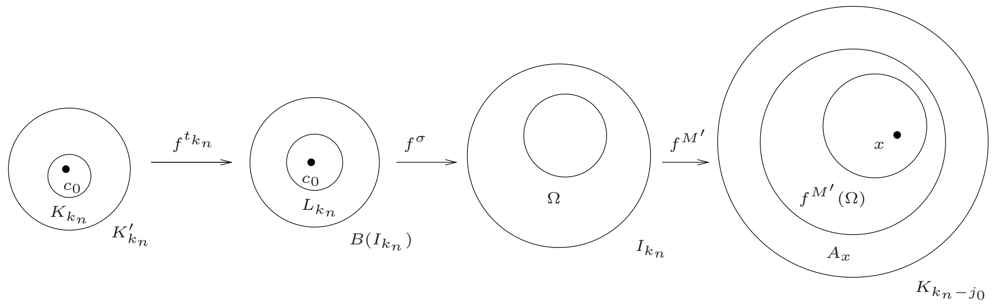

Figure 12

By Claim 2 and (F1) in section 3,

$$\begin{aligned}
\text{mod } (K_{k_n-j_0}-\overline{A_x}) &\leqslant \quad \text{mod } (K_{k_n-j_0}-\overline{f^{M'}(\Omega)}) \\
&\leqslant \deg(f^{t_{k_n}+\sigma+M'}|_{K'_{k_n}})\cdot \quad \text{mod } (K'_{k_n}-\overline{K_{k_n}}) \\
&= \deg(f^{t_{k_n}}|_{K'_{k_n}})\deg(f^{\sigma}|_{B(I_{k_n})})\deg(f^{M'}|_{I_{k_n}})\cdot\mu_{k_n} \\
&= \deg(f^{t_{k_n}}|_{K_{k_n}})\deg(f^{\sigma}|_{B(I_{k_n})})\deg(f^{M'}|_{I_{k_n}})\cdot\mu_{k_n} \\
&\leqslant d_{max}^{b^2}\cdot d_{max}^{b^2}\cdot d_{max}^{b^2}d_1^{j_0-1}\cdot\mu_{k_n} \\
&= d_3\mu_{k_n},
\end{aligned}$$

where $d_3 = d_{max}^{3b^2}d_1^{j_0-1}$ is a constant depending only on b and d_{max}. See Figure 12.

Take $\eta = d_3^{-1}{d_{max}}^{-(b-1)}$. We have

$$\text{mod } (B'_y-\overline{B_y}) \geqslant \eta \ \text{mod } (K_{k_n-j_0}-\overline{A_x}).$$

□

Now we have a holomorphic proper mapping $f^M : (K_{k_n-j_0}, A'_x, A_x) \to (K_{k_n-N}, B'_y, B_y)$ satisfying

(1) $d_0^{(3b+2)(N-j_0)} \leqslant D = \deg(f^M|_{K_{k_n-j_0}}) \leqslant d_1^{N-j_0}$,

(2) $\deg(f^M|_{A'_x}) = \deg(f^M|_{A_x}) \leqslant d_2$,

(3) $\text{mod } (B'_y-\overline{B_y}) \geqslant \eta\, \text{mod } (K_{k_n-j_0}-\overline{A_x})$,

where $d_0 = \deg_{c_0} f$ and d_1, d_2, η are constants depending only on b and d_{max}. See Figure 13.

By the Kahn-Lyubich Lemma,

$$\text{mod } (K_{k_n-N}-\overline{B_y}) \leqslant C\eta^{-1}d_2^2 \ \text{mod } (K_{k_n-j_0}-\overline{A_x}) \tag{4.1}$$

or

$$\text{mod } (K_{k_n-j_0}-\overline{A_x}) \geqslant \epsilon. \tag{4.2}$$

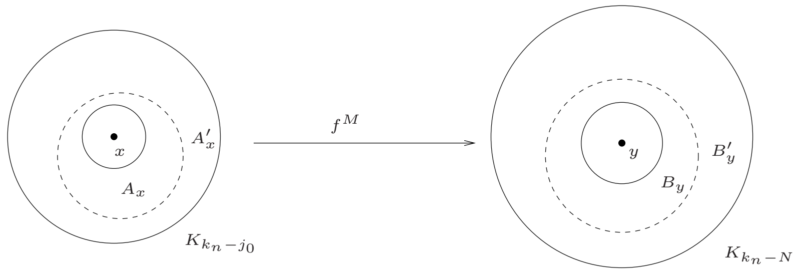

Figure 13

We first prove that the inequality (4.1) is impossible for N large enough. For each $j_0 \leqslant i \leqslant N-1$, let $V_i(y) = L_y(K_{k_n-i})$, $V_i'(y) = L_y(K'_{k_n-i})$, and $f^{r_i}(V_i(y)) = K_{k_n-i}$. See Figure 14. Then $f^{r_i}(V_i'(y)) = K'_{k_n-i}$ and

$$\deg(f^{r_i}|_{V_i(y)}) = \deg(f^{r_i}|_{V_i'(y)}) \leqslant {d_{max}}^b.$$

Therefore

$$\begin{aligned}\text{mod } (V_i'(y) - \overline{V_i(y)}) &= \frac{1}{\deg(f^{r_i}|_{V_i(y)})} \text{ mod } (K'_{k_n-i} - \overline{K_{k_n-i}}) \\ &\geqslant {d_{max}}^{-b}\mu_{k_n-i} \\ &\geqslant {d_{max}}^{-b}\mu_{k_n}\end{aligned}$$

and

$$\text{mod } (K_{k_n-N} - \overline{B_y}) \geqslant (N-j_0){d_{max}}^{-b}\mu_{k_n}.$$

By the proof of Claim 3,

$$\text{mod } (K_{k_n-j_0} - \overline{A_x}) \leqslant d_3\mu_{k_n}.$$

Hence

$$\text{mod } (K_{k_n-N} - \overline{B_y}) \geqslant (N-j_0)d_3^{-1}d_{max}^{-b} \text{ mod } (K_{k_n-j_0} - \overline{A_x}).$$

This implies that the inequality (4.1) is impossible for N large enough.

Take a large N_0 such that (4.1) does not hold. We have

$$\text{mod } (K_{k_n-j_0} - \overline{A_x}) \geqslant \epsilon > 0,$$

where ϵ depends only on η and N_0. This contradicts the fact

$$\text{mod } (K_{k_n-j_0} - \overline{A_x}) \leqslant d_3\mu_{k_n} \to 0$$

as $n \to \infty$. This completes the proof of Lemma 3. $\square$

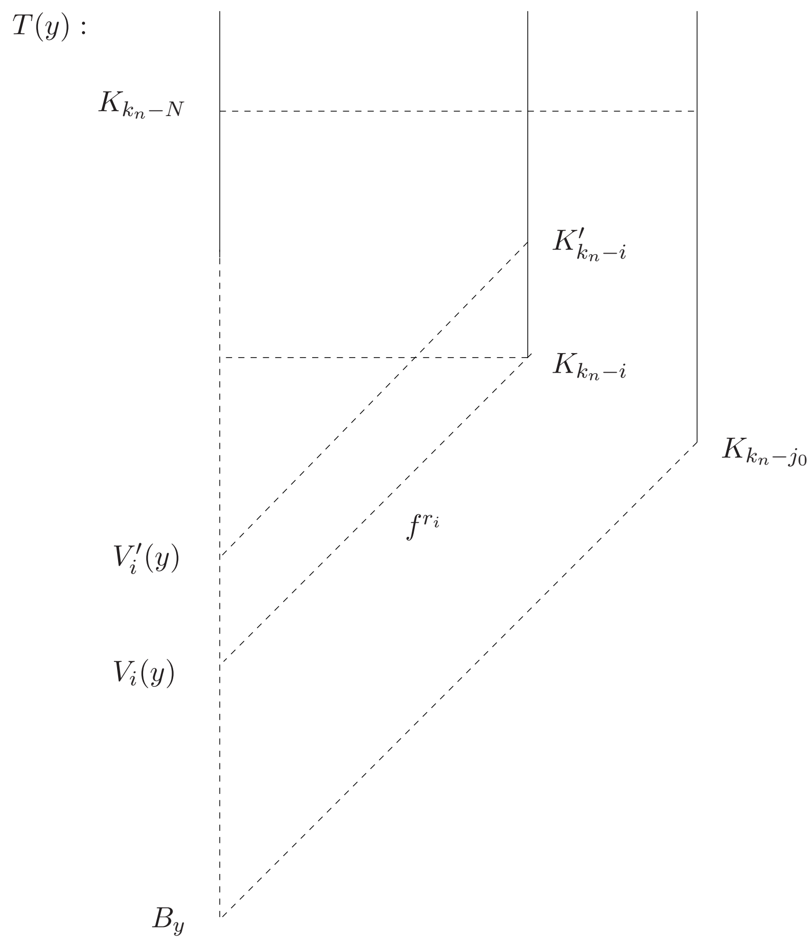

Figure 14

Proof of the Main Proposition

By Lemma 3, $\mu_{k_n} \geqslant \mu > 0$ for some constant μ. The Grötzsch's inequality implies that

$$\text{mod }(P_0(c_0) - K_f(c_0)) \geqslant \sum_{k=0}^{\infty} \text{ mod }(K'_{k_n} - \overline{K_{k_n}}) = +\infty$$

and $K_f(c_0) = \bigcap_{n \geqslant 0} P_n(c_0) = \{c_0\}$.

For any $x \in K_f$ with $x \to c_0$, let $V_n(x) = L_x(K_{k_n})$, $f^{r_n}(V_n(x)) = K_{k_n}$, and let $V'_n(x) = \mathrm{Comp}_x f^{-r_n}(K'_{k_n})$. The degree of $f^{r_n} : V_n(x) \to K_{k_n}$ is uniformly bounded for all n.

If there are infinitely many n, say $\{n_j\}$, such that there is at most one piece in $\{V'_n(x), f(V'_n(x)), \ldots, f^{r_n}(V'_n(x)) = K'_{k_n}\}$ containing c for any $c \in \mathrm{Crit} - [c_0]$, then the degree of $f^{r_{n_j}} : V'_{n_j}(x) \to K'_{k_{n_j}}$ is uniformly bounded

for all j and there is a constant $\widetilde{\mu} > 0$ such that

$$\mathrm{mod}\,(V'_{n_j}(x) - \overline{V_{n_j}(x)}) \geqslant \widetilde{\mu}$$

for all j. In this case, $K_f(x) = \{x\}$.

Suppose for each large n, there are two pieces in

$$\{V'_n(x), f(V'_n(x)), \ldots, f^{r_n}(V'_n(x)) = K'_{k_n}\}$$

containing c for some $c \in \mathrm{Crit} - [c_0]$. There exist $c_1 \in \mathrm{Crit} - [c_0]$ and a subsequence $\{n_j\}$ such that there are two pieces in

$$\{V'_{n_j}(x), f(V'_{n_j}(x)), \ldots, f^{r_{n_j}}(V'_{n_j}(x)) = K'_{k_{n_j}}\}$$

containing c_1. See Figure 15.

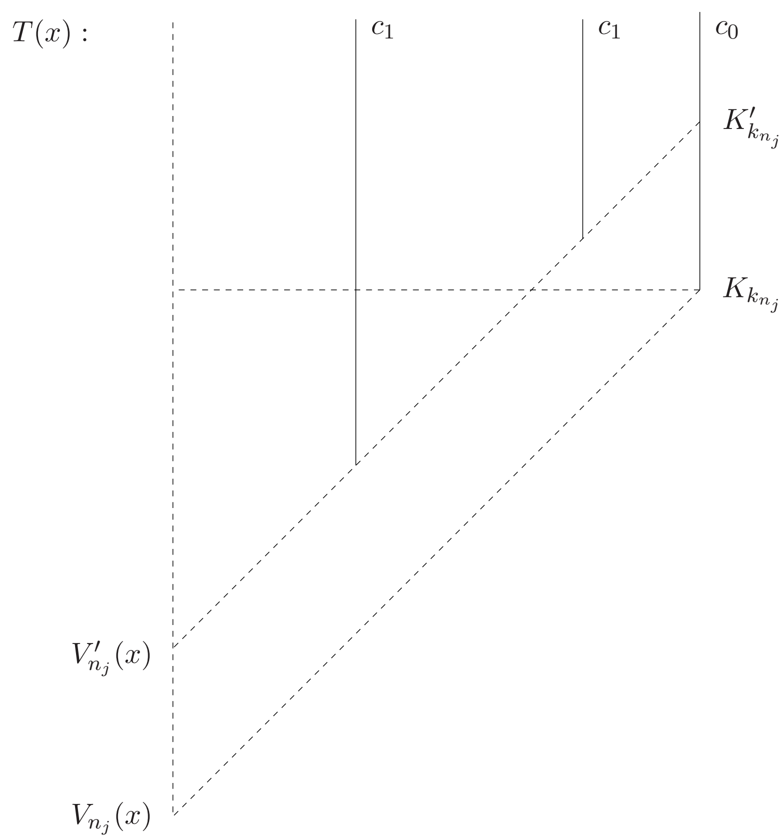

Figure 15

We conclude that $c_1 \to c_1$ and $c_1 \to c_0$. From the conditions $(K'_{k_{n_j}} - K_{k_{n_j}}) \cap \mathrm{orb}([c_0]) = \emptyset$ and (T3), we have $c_1 \notin [c_0]$. The Lemma 1 implies $T(c_1)$ is reluctantly recurrent. By Proposition 1, $K_f(x) = \{x\}$ since $x \to c_1$.

This ends the proof of the Main Proposition. $\square$

5 Wandering components of filled-in Julia sets

In this section, let f be an arbitrary complex polynomial with disconnected Julia set(without the assumption that each critical component of the filled-in Julia set is aperiodic). We will prove that each wandering component of K_f is a point. It concludes that all but countably many components of the filled-in Julia set are single points. This result gives an affirmative answer of a question in a remark in Milnor's book, see[28].

For any cubic polynomial with disconnected Julia set, it follows from Theorem 5.2 and Theorem 5.3 in [4] that each wandering component of the filled-in Julia set is a point.

For a polynomial with high degree, the second author proved

Theorem D ([36]). *Let f be a complex polynomial providing each critical point c in wandering Julia components(if any) is non-recurrent, i.e.,$c \notin \omega(c)$. Then each wandering component of the filled-in Julia set K_f is a point.*

For any $x \in K_f$, let

$$\mathrm{Crit}(x) = \{c \in \mathrm{Crit} \mid x \to c\},$$

where Crit is the set of critical points in the filled-in Julia set.

Let

$$\begin{aligned}
&\mathrm{Crit}_{\mathrm{n}}(x) = \{c \in \mathrm{Crit}(x) \mid T(c) \text{ is non-critical}\},\\
&\mathrm{Crit}_{\mathrm{p}}(x) = \{c \in \mathrm{Crit}(x) \mid T(c) \text{ is persistently recurrent}\},\\
&\mathrm{Crit}_{\mathrm{r}}(x) = \{c \in \mathrm{Crit}(x) \mid T(c) \text{ is reluctantly recurrent}\},\\
&\mathrm{Crit}_{\mathrm{en}}(x) = \{c' \in \mathrm{Crit}(x) \mid c' \not\to c' \text{ and } c' \to c \text{ for some } c \in \mathrm{Crit}_{\mathrm{n}}(x)\},\\
&\mathrm{Crit}_{\mathrm{ep}}(x) = \{c' \in \mathrm{Crit}(x) \mid c' \not\to c' \text{ and } c' \to c \text{ for some } c \in \mathrm{Crit}_{\mathrm{p}}(x)\},\\
&\mathrm{Crit}_{\mathrm{er}}(x) = \{c' \in \mathrm{Crit}(x) \mid c' \not\to c' \text{ and } c' \to c \text{ for some } c \in \mathrm{Crit}_{\mathrm{r}}(x)\}.
\end{aligned}$$

Then

$$\mathrm{Crit}(x) = \mathrm{Crit}_{\mathrm{n}}(x) \cup \mathrm{Crit}_{\mathrm{p}}(x) \cup \mathrm{Crit}_{\mathrm{r}}(x) \cup \mathrm{Crit}_{\mathrm{en}}(x) \cup \mathrm{Crit}_{\mathrm{ep}}(x) \cup \mathrm{Crit}_{\mathrm{er}}(x).$$

Proposition 2. *Suppose $x \in K_f$ and $x \not\to c$ for any critical point c contained in a periodic component of the filled-in Julia set K_f. Then $K_f(x) = \{x\}$.*

Proof. If $\mathrm{Crit}(x) = \emptyset$, then $T(x)$ is non-critical. By the Proposition 1(1), we have $K_f(x) = \{x\}$.

If $\mathrm{Crit}_{\mathrm{n}}(x) \cup \mathrm{Crit}_{\mathrm{r}}(x) \neq \emptyset$, by the same methods as in the proof of Proposition 1(2), we have $K_f(x) = \{x\}$.

Now we suppose that

$$\mathrm{Crit}(x) = \mathrm{Crit}_{\mathrm{p}}(x) \cup \mathrm{Crit}_{\mathrm{ep}}(x) \neq \emptyset.$$

Since $x \not\to c$ for any critical point c contained in a periodic component of the filled-in Julia set K_f, hence $T(c)$ is not periodic for any $c \in \mathrm{Crit}(x)$. By the proof in the Main Proposition, we have $K_f(x) = \{x\}$. $\square$

We state a result stronger than the Main Theorem as the following

Theorem. *Let f be a polynomial of degree $d \geqslant 2$ with a disconnected Julia set and let K be a connected component of the filled-in Julia set K_f.*

(1) If $f^n(K)$ is a periodic component for some $n \geqslant 0$ and there is at least one critical point in the cycle of this component, then K is not a point.

(2) If $f^n(K)$ is a periodic component for some $n \geqslant 0$ and there is no critical points in the cycle of this component, then K is a point.

(3) If K is a wandering component, i.e., $f^n(K)$ is not periodic for all $n \geqslant 0$, then K is a point.

Proof. The proofs of (1) and (2) are routine, see [4].

By iteration, we may assume that each periodic component containing critical points(if any) is invariant. Let K be a wandering component of K_f and x be a point in K. Then $K = K_f(x) = \bigcap_{k\geqslant 0} P_k(x)$. There are two possibilities

(a) There is a critical point c_0 contained in an invariant component of the filled-in Julia set K_f such that $x \to c_0$.

(b) $x \not\to c$ for any critical point c contained in an invariant component of the filled-in Julia set K_f.

In case (a), let $l_k \geqslant 1$ be the first moment such that $f^{l_k}(x) \in P_k(c_0)$ for any $k \geqslant 0$, i.e., (k, l_k) is the first c_0-position on the k-th row in the tableau $T(x)$. Then there is an integer $D \geqslant 1$ such that $\deg(f^{l_k} : \ P_{k+l_k}(x) \to P_k(f^{l_k}(x))) \leqslant D$ for all k. Since $K = K_f(x)$ is wandering, there exists an integer $n_k > k$ such that $(n_k - 1, l_k)$ is a c_0-position and (n_k, l_k) is not critical. By the tableau rule (T3) in section 2, there is no critical position on the diagonal

$$\{(n, m) \mid n + m = n_k + l_k, \quad 1 \leqslant n \leqslant n_k\}.$$

Then

$$\begin{aligned}\deg(f^{n_k+l_k} : \ P_{n_k+l_k}(x) \to P_0(f^{n_k+l_k}(x))) &= \deg(f^{l_k} : \ P_{n_k+l_k}(x) \to P_{n_k}(f^{l_k}(x))) \\ &\leqslant \deg(f^{l_k} : \ P_{k+l_k}(x) \to P_k(f^{l_k}(x))) \\ &\leqslant D.\end{aligned}$$

There is a positive constant ν such that

$$\mathrm{mod}\ (P_{n_k+l_k}(x) - \overline{P_{n_k+l_k+1}(x)}) \geqslant \nu$$

for all $k \geqslant 0$. This implies that $K = K_f(x) = \{x\}$ is a point.

In case (b), it follows from Proposition 2 that $K = K_f(x) = \{x\}$. $\square$

An immediately consequence is

Corollary. *Let f be a polynomial of degree $d \geqslant 2$ with a disconnected Julia set. Then all but countably many components of the filled-in Julia set are single points.*

Remark. This corollary is not true for arbitrary rational maps, see [26].

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Weiyuan Qiu
School of Mathematical Sciences
Fudan University
Shanghai, 200433
P.R.China
wyqiu@fudan.edu.cn

Yongcheng Yin

Department of Mathematics
Zhejiang University University
Hangzhou, 310027
P.R.China
yin@zju.edu.cn